ESC103 Material - F2025

Vectors, Matrices, Linear Systems, and ODEs

Units 1–20 + Exam Review + Ben's Problem-Solving Toolkit from Midterm

Contents

Problem-Solving Toolkit

Big Picture

ESC103 problems almost always reduce to one of these patterns:

- **Vector geometry:** lengths, angles, projections, lines, planes.
- **Linear systems:** interpreting $A\mathbf{x} = \mathbf{b}$ using row and column pictures.
- **Matrix structure:** independence, rank, null space, factorizations like $A = CR$.
- **Solving systems:** Gaussian elimination, inverses.
- **Best approximation:** least squares (projecting onto a column space).
- **Dynamics:** ODEs written as vector equations; Euler / Improved Euler

Default Strategy

When you see a problem:

1. **Identify the object.** Is it a vector, line, plane, matrix, system, or ODE?
2. **Pick a picture.**
   - Geometry: sketch vectors / lines / planes.
   - Row picture: each row is a line/plane.
   - Column picture: $A\mathbf{x}$ as combination of column vectors.
3. **Use the right tool.**
   - Dot / cross products and projections for geometry.
   - Gaussian elimination or RREF for systems.
   - $A = CR$ or column space / null space for structure questions.
   - Normal equations for least squares.
   - Euler / Improved Euler for ODEs.

Practice: Toolkit Drills

1. Given a random system of equations, practice writing:
   - the scalar form,
   - the matrix form $A\mathbf{x} = \mathbf{b}$,
   - the row picture,
   - the column picture.
2. For each unit below, pick **one** example and redraw the geometric picture completely from memory.

1 Unit 1: Vectors in $\mathbb{R}^n$

1.1 What is a vector?

Vector = Arrow + Numbers

A vector in $\mathbb{R}^n$ is:

- geometrically: an arrow with direction and length;
- algebraically: a column of n numbers.

For example, in $\mathbb{R}^3$,

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

points 2 units in x, -1 in y, 4 in z from the origin.

1.2 Basic operations

- **Addition:** tip-to-tail picture.
- **Scalar multiplication:** stretch or flip the arrow.

The magnitude (length) of $\mathbf{v} = \begin{bmatrix} v_1 \\ \dots \\ v_n \end{bmatrix}$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Practice: Unit 1 Practice

1. Sketch $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$. Draw $\mathbf{u} + \mathbf{v}$.
2. Compute the length of $\mathbf{w} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and verify using the picture.
3. In $\mathbb{R}^3$, describe geometrically the linear combination

$$\left\{ \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} : \lambda \in \mathbb{R} \right\}.$$

2 Unit 2: Dot Product and Projection

2.1 Dot product

Algebra and Geometry

For $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n.$$

Geometrically,

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta,$$

where θ is the angle between them.

Warning: Orthogonality

If $\mathbf{v} \cdot \mathbf{w} = 0$ and neither vector is zero, that means they are *perpendicular*. In our course questions, *"orthogonal"* always means that the dot product is zero.

2.2 Projection onto a line

When you project $\mathbf{w}$ onto the line spanned by $\mathbf{v}$ you get

$$\operatorname{proj}_{\mathbf{v}} \mathbf{w} = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

Example: Projection in $\mathbb{R}^2$

Let

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Then

$$\mathbf{w} \cdot \mathbf{v} = 3 \cdot 2 + 4 \cdot 1 = 10, \quad \mathbf{v} \cdot \mathbf{v} = 2^2 + 1^2 = 5.$$

So

$$\operatorname{proj}_{\mathbf{v}} \mathbf{w} = \frac{10}{5} \mathbf{v} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Practice: Unit 2 Practice

1. Compute $\mathbf{v} \cdot \mathbf{w}$ and the angle between $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
2. Find $\operatorname{proj}_{\mathbf{v}} \mathbf{w}$ for $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.
3. Given two nonzero vectors with dot product zero, draw a picture and explain why the Pythagorean theorem appears naturally.

3 Unit 3: Cross Product in $\mathbb{R}^3$

3.1 Definition and right-hand rule

For $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$,

$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ -(v_1 w_3 - v_3 w_1) \\ v_1 w_2 - v_2 w_1 \end{bmatrix}.$$

Geometric Meaning

- $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.
- $\|\mathbf{v} \times \mathbf{w}\|$ is the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.
- Right-hand rule: rotate from $\mathbf{v}$ to $\mathbf{w}$; your thumb points along $\mathbf{v} \times \mathbf{w}$.

Practice: Unit 3 Practice

1. Compute $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ and interpret.
2. Show that $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if $\mathbf{v}$ and $\mathbf{w}$ are parallel.
3. For generic nonparallel $\mathbf{v}$ and $\mathbf{w}$, sketch the parallelogram, label its area, and mark the vector $\mathbf{v} \times \mathbf{w}$.

4 Unit 4: Lines in 3D

4.1 Vector equation of a line

Line = Point + Direction

A line in $\mathbb{R}^3$ is completely determined by:

- one point on the line $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$;
- a direction vector $\mathbf{d}$ parallel to the line.

All points on the line can be written as

$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{d}, \quad t \in \mathbb{R}.$$

Component-wise,

$$x = x_0 + d_1 t, \quad y = y_0 + d_2 t, \quad z = z_0 + d_3 t.$$

Practice: Unit 4 Practice

1. Find the vector equation of the line through $P = (1, -2, 5)$ and $Q = (4, 0, 9)$.
2. Check whether the point $(7, 2, 17)$ lies on that line.
3. Give a geometric explanation of why a single direction vector is enough to describe the whole line.

5 Unit 5: Planes in 3D

5.1 Vector and scalar equations

Plane = Point + Two Directions

A plane can be described by:

- a base point $\mathbf{r}_0$;
- two non-parallel direction vectors $\mathbf{d}_1, \mathbf{d}_2$.

Any point on the plane has the form

$$\mathbf{r} = \mathbf{r}_0 + s\,\mathbf{d}_1 + t\,\mathbf{d}_2, \quad s, t \in \mathbb{R}.$$

Another way: use a *normal vector* $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ perpendicular to the plane. Then any point (x, y, z) on the plane satisfies

$$ax + by + cz = d$$

for some constant d.

Example: Plane Through a Point

Suppose a plane has normal $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ and passes through $(1, 0, 0)$. Then

$$2 \cdot 1 - 3 \cdot 0 + 0 \cdot 0 = 2,$$

so its scalar equation is

$$2x - 3y + 0z = 2.$$

Practice: Unit 5 Practice

1. A plane is described by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

   Find a normal vector and then the scalar equation.

2. For $2x - 3y = 0$, explain why this is
   - a line in $\mathbb{R}^2$,
   - a plane in $\mathbb{R}^3$.

3. Describe the intersection of two non-parallel planes in $\mathbb{R}^3$.

6 Unit 6: Row Picture and Column Picture

6.1 Matrix form of a system

A system of linear equations can be written as $A\mathbf{x} = \mathbf{b}$.

Two Mental Pictures

- **Row picture:** each row is a line/plane; solutions are intersections.
- **Column picture:** $A\mathbf{x}$ is a linear combination of columns; we ask whether $\mathbf{b}$ is in the span of the columns.

Example: Simple System

$$\begin{cases} x - 2y = 1, \\ 3x + 2y = 11. \end{cases}$$

Row picture: two lines in the plane intersecting at $(3, 1)$.
Column picture:

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

We are finding the right linear combination of the columns to produce the right-hand side.

Practice: Unit 6 Practice

1. Write $2x + y = 3$, $4x - y = 1$ in matrix form and sketch the row picture.
2. For your matrix A, write the column picture equation $A\mathbf{x} = \mathbf{b}$.
3. Interpret "no solution" in the column picture: where is $\mathbf{b}$ relative to $\mathcal{C}(A)$?

7 Unit 7: Matrix-Vector Multiplication and Basic Matrix Ops

7.1 What does $A\mathbf{x}$ really mean?

Let A be $m \times n$ and $\mathbf{x} \in \mathbb{R}^n$.

Two Equivalent Views

- **Row view:** each entry of $A\mathbf{x}$ is a dot product of a row with $\mathbf{x}$.
- **Column view:** $A\mathbf{x}$ is a linear combination of columns with coefficients from $\mathbf{x}$.

7.2 Matrix addition and scalar multiplication

Matrices of the same size add entrywise:

$$(A + B)_{ij} = a_{ij} + b_{ij}, \qquad (cA)_{ij} = c\, a_{ij}.$$

Practice: Unit 7 Practice

1. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Compute $A\mathbf{x}$ by the row-dot-product view and by the column-combination view.
2. For $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, sketch the effect of A on the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$.
3. Show that matrix addition is commutative but matrix multiplication is generally not.

8 Unit 8: Linear Independence, Column Space, and Rank

8.1 Independence

Informal Idea

Vectors are *linearly independent* if none of them is "redundant". You cannot build one as a linear combination of the others.

Formally, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is independent if

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = \cdots = c_k = 0.$$

8.2 Column space and rank

Column Space and Rank

- $\mathcal{C}(A)$: all linear combinations of the columns of A.
- $\text{rank}(A)$: number of linearly independent columns of A = dimension of $\mathcal{C}(A)$.

Practice: Unit 8 Practice

1. Check whether $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ are independent.
2. For $A = \begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\1 & 3 & 5\end{bmatrix}$, find $\text{rank}(A)$ and give a basis for $\mathcal{C}(A)$.
3. Geometrically describe the column space in $\mathbb{R}^3$ when $\text{rank}(A) = 1$, $\text{rank}(A) = 2$, and $\text{rank}(A) = 3$.

9 Unit 9: Matrix Multiplication

9.1 Definition and pictures

If A is $m \times n$ and B is $n \times p$, then $C = AB$ is $m \times p$ and

$$c_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B).$$

Column Picture

The j-th column of AB is A times the j-th column of B:

$$(AB)_j = A\,\mathbf{b}_j.$$

So multiplying by B first forms some combinations, then A acts on those results.

Practice: Unit 9 Practice

1. Compute AB for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ and verify $AB \neq BA$.
2. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, explain geometrically what multiplying by A does in $\mathbb{R}^2$.
3. Show that $A(I\mathbf{x}) = (AI)\mathbf{x} = A\mathbf{x}$ for a compatible identity matrix I.

10 Unit 10: Factoring $A = CR$

10.1 Why factor a matrix?

Idea of $A = CR$

- C collects the *independent columns* of A (a basis for $\mathcal{C}(A)$).
- R records the coefficients that reconstruct every column of A from columns of C.

This splits "geometry" (column space) from "instructions" (how to build each column).

10.2 Constructing C and R

1. Scan columns of A from left to right; keep the first nonzero, then each column that cannot be written as a combination of previous ones. These form C.
2. For each original column $\mathbf{a}_j$ write $\mathbf{a}_j = r_{1j}\mathbf{c}_1 + \cdots + r_{rj}\mathbf{c}_r$. The coefficients $(r_{1j}, \ldots, r_{rj})$ form the j-th column of R.

Practice: Unit 10 Practice

1. Factor $A = \begin{bmatrix} 2 & 6 & 4 \\ 4 & 12 & 8 \\ 1 & 3 & 2 \end{bmatrix}$ as $A = CR$.
2. From your factorization, read off $\mathcal{C}(A)$ and $\operatorname{rank}(A)$.
3. What happens to R if all columns of A are independent?

11 Unit 11: Solving $A\mathbf{x} = \mathbf{0}$ and Null Space

11.1 Null space and nullity

Null Space

The *null space* of A is

$$\mathcal{N}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}.$$

It consists of all input vectors that A **collapses to zero**. When solving $A\mathbf{x} = 0$, the free variables represent the entire null space.

Rank–Nullity

For an $m \times n$ matrix,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Here, nullity(A) is the **dimension** of the null space (the number of free variables). n represents the number of columns in A

11.2 Using $A = CR$

If $A = CR$ with independent columns in C,

$$A\mathbf{x} = \mathbf{0} \iff CR\mathbf{x} = \mathbf{0} \iff R\mathbf{x} = \mathbf{0}.$$

So you can work with the smaller system $R\mathbf{x} = \mathbf{0}$.

Practice: Unit 11 Practice

1. Find a basis for $\mathcal{N}(A)$ where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix}$.
2. For a 4×5 matrix with rank 3, how many free variables does $A\mathbf{x} = \mathbf{0}$ have? What is nullity(A)?
3. Explain geometrically what $\mathcal{N}(A)$ looks like in $\mathbb{R}^3$ when rank$(A) = 1$ and when rank$(A) = 2$.

12 Unit 12: Elementary Row Operations and Elementary Matrices

12.1 Three legal row moves

Elementary Row Operations

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another.

These are the only moves allowed in Gaussian elimination.

Each row operation corresponds to left-multiplication by an *elementary matrix* built from the identity by the same row move.

Practice: Unit 12 Practice

1. Find the elementary matrix that swaps rows 1 and 3 of a 3×3 matrix.
2. Find the elementary matrix that performs $R_2 \leftarrow R_2 - 4R_1$ on a 2×2 matrix.
3. Explain why elementary row operations do not change the solution set of $A\mathbf{x} = \mathbf{b}$.

13 Unit 13: Gaussian Elimination, REF and RREF

13.1 Echelon forms

REF vs RREF

- **REF** (row echelon form): staircase of leading nonzero entries moving right as you go down.
- **RREF** (reduced REF): in addition, each leading entry is 1 and is the only nonzero entry in its column.

Gaussian elimination:

1. Forward elimination: turn A into REF.
2. Optional back substitution or further elimination to get RREF.

Practice: Unit 13 Practice

1. Row-reduce $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 1 & 3 & 0 \end{bmatrix}$ to RREF.
2. From your RREF, identify $\operatorname{rank}(A)$ and a basis for $\mathcal{N}(A)$.
3. Describe the solution set of $A\mathbf{x} = \mathbf{0}$ in words (point, line, plane, etc.).

14 Unit 14: Solving $A\mathbf{x} = \mathbf{b}$

14.1 Augmented matrix

To solve $A\mathbf{x} = \mathbf{b}$, work with the augmented matrix

$$[A \,|\, \mathbf{b}] \longrightarrow [R_0 \,|\, \mathbf{d}],$$

where R_0 is RREF.

Types of Solution Sets

- **No solution:** a row of zeros in R_0 with a nonzero entry in $\mathbf{d}$.
- **Unique solution:** pivot in every variable column.
- **Infinitely many solutions:** at least one free variable, no contradictions.

Practice: Unit 14 Practice

1. Solve $\begin{cases} x + 2y - z = 1, \\ 2x + 4y - 2z = 2, \\ x + 3y - z = 2. \end{cases}$ and classify the solution set.
2. Explain how the column picture tells you whether $A\mathbf{x} = \mathbf{b}$ has a solution.
3. Construct a 3×3 system with infinitely many solutions and write one free parameter explicitly.

15 Unit 15: Rank Nullity

15.1 Rank and the column space

Rank

For an $m \times n$ matrix A:

- The *column space* $\mathcal{C}(A)$ is the set of all linear combinations of the columns of A.
- The *rank* of A, written $\text{rank}(A)$, is the dimension of $\mathcal{C}(A)$.

Geometrically, $\text{rank}(A)$ is:

- 0 if all columns are zero,
- 1 if all columns lie on a single line through the origin,
- 2 if the columns span a plane through the origin (in $\mathbb{R}^3$),
- etc.

15.2 Null space, nullity, and the rank–nullity theorem

Null Space and Nullity

The *null space* of A is

$$\mathcal{N}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}.$$

It contains all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$.
The *nullity* of A, written $\text{nullity}(A)$, is the dimension of $\mathcal{N}(A)$ (the number of free variables in $A\mathbf{x} = \mathbf{0}$).

Rank–Nullity Theorem

For an $m \times n$ matrix A,

$$\text{rank}(A) + \text{nullity}(A) = n,$$

where n is the number of columns of A.

15.3 Consistency of $A\mathbf{x} = \mathbf{b}$

Consistency Conditions

Consider $A\mathbf{x} = \mathbf{b}$.

- The system is **consistent** (has at least one solution) if $\mathbf{b} \in \mathcal{C}(A)$.
- In RREF, inconsistency appears as a row

$$\begin{bmatrix} 0 & 0 & \dots & 0 \mid d \end{bmatrix}, \quad d \neq 0.$$

- If the system is consistent and every column corresponding to a variable has a pivot, the solution is unique.
- If the system is consistent and at least one variable column has no pivot, the system has infinitely many solutions (free variables).

Practice: Unit 15 Practice

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}.$$

   Row-reduce A to RREF, find $\operatorname{rank}(A)$ and a basis for $\mathcal{C}(A)$.

2. For the same matrix A, find a basis for $\mathcal{N}(A)$ and compute $\operatorname{nullity}(A)$. Check that $\operatorname{rank}(A) + \operatorname{nullity}(A)$ equals the number of columns.

3. Consider the system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

   Use the augmented matrix and RREF to decide:

   - Is the system consistent?
   - If yes, does it have a unique solution or infinitely many?

16 Unit 16: Least Squares and Best-Fit Lines

16.1 When no exact solution exists

Sometimes $A\mathbf{c} = \mathbf{y}$ has no solution. We then look for the vector $\mathbf{c}_{\text{ls}}$ that makes $\|A\mathbf{c} - \mathbf{y}\|$ as small as possible.

Normal Equations

Least-squares solutions satisfy

$$A^T A\,\mathbf{c}_{\text{ls}} = A^T\mathbf{y}.$$

If $A^T A$ is invertible, then

$$\mathbf{c}_{\text{ls}} = (A^T A)^{-1} A^T\mathbf{y}.$$

Geometric picture: $A\mathbf{c}_{\text{ls}}$ is the projection of $\mathbf{y}$ onto $\mathcal{C}(A)$.

16.2 Best-fit line

For data points (x_i, y_i) and model $y = a + bx$,

$$A = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

Practice: Unit 16 Practice

1. Given data points $(0, 4)$, $(2, -1)$, $(3, 0)$, compute the best-fit line.
2. Compute the residual vector $\mathbf{e} = \mathbf{y} - A\mathbf{c}_{\text{ls}}$ and its norm.
3. Explain why $\mathbf{e}$ is orthogonal to every column of A.

17 Unit 17:Euler's

17.1 Turning $y'' = -y$ into a first-order system

State Vector

Second–order equations like $y'' = -y$ cannot be used directly in Euler's method, because Euler requires a first-order system. To fix this, we introduce a new vector that stores both y and y'. Let

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}.$$

Here $z_1 = y$ and $z_2 = y'$.
Differentiate each component:

$$z_1' = y' = z_2, \qquad z_2' = y'' = -y = -z_1.$$

This gives a first-order system:

$$\mathbf{z}' = A\mathbf{z}, \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

17.2 Euler's method

To approximate the solution forward in time, we use the update rule

$$\mathbf{z}_{n+1} = \mathbf{z}_n + A\mathbf{z}_n\, \Delta t.$$

Euler uses the slope $A\mathbf{z}_n$ at the current point and takes a small step of size Δt in that direction. Because $\mathbf{z} = (y, y')$, each step updates both the position y and the velocity y'.

Practice: Unit 17 Practice

1. Write the system $\mathbf{z}' = A\mathbf{z}$ explicitly for $y'' = -y$ (two scalar first-order equations).
2. Starting from $y(0) = 0$, $y'(0) = 1$, take one Euler step with $\Delta t = \pi/4$ to approximate $y(\pi/4)$; compare with the exact value $\sin(\pi/4)$.
3. Explain why smaller Δt typically improves Euler's accuracy.

18 Unit 18: Higher Order Systems and Improved Euler

18.1 From higher order to systems

Any higher-order linear ODE can be rewritten as a first-order system by introducing extra variables for each derivative:

$$y^{(k)} = \text{new variable}.$$

For example, $y'' + 4y = 0$ can be written in terms of

$$z_1 = y, \quad z_2 = y'$$

so that

$$z_1' = z_2, \qquad z_2' = -4z_1.$$

This gives $\mathbf{z}' = A\mathbf{z}$ for a suitable matrix A, and we can apply EM methods to $\mathbf{z}$.

18.2 Improved Euler Method

Improved Euler

The Improved Euler (or Heun) method is a **predictor–corrector** method. It improves on ordinary Euler by using two slope estimates instead of just one.

Step 1: Predict Take one forward Euler step using the slope at the beginning:

$$\mathbf{z}^{\text{E}}_{n+1} = \mathbf{z}_n + A\mathbf{z}_n \, \Delta t.$$

Step 2: Correct Compute the slope at this predicted point and average:

$$\mathbf{z}^{\text{IE}}_{n+1} = \mathbf{z}_n + \frac{1}{2}\big(A\mathbf{z}_n + A\mathbf{z}^{\text{E}}_{n+1}\big)\Delta t.$$

Geometric idea: Euler uses only the initial slope, which can drift away from the true curve. Improved Euler "looks ahead/" by predicting the next point, checking the slope there, and then averaging the two slopes to get a better estimate.

Practice: Unit 18 Practice

1. Apply Improved Euler with one step of size $\Delta t = \pi/4$ to approximate $\sin(\pi/4)$ for $y'' = -y$.
2. Compare numerically the Euler and Improved Euler approximations at $t = \pi/4$.
3. Explain the geometric reason the average slope is more accurate than the single slope used by Euler.

19 Unit 19: Boundary Value Problems

19.1 Finite differences for y''

For equally spaced points $t_n = t_0 + n\Delta t$,

$$y''(t_n) \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{(\Delta t)^2}.$$

From ODE to Linear System

For an ODE like $y'' + y = 0$ with boundary conditions $y(t_0) = \alpha$, $y(t_m) = \beta$:

- Approximate y'' at interior points using finite differences.
- This gives linear equations in $y_1, \ldots, y_{m-1}$.
- Collect them into a matrix system $K\mathbf{y} = \mathbf{f}$.

Solving this linear system gives approximate values for the solution at the grid points.

Practice: Unit 19 Practice

1. For $y'' + y = 0$, $y(0) = 0$, $y(\pi/2) = 1$, use $\Delta t = \pi/4$ (two interior points) to set up and solve the system.
2. For the same problem, use $\Delta t = \pi/8$ (four intervals) and write the 3×3 system for y_1, y_2, y_3, use this to solve.
3. Sketch the approximate solutions for both step sizes and compare to $y(t) = \sin t$.

20 Unit 20: Inverse Matrices (Square Matrices)

20.1 Definition and meaning

Inverse Matrix

In this course we only work with square (2×2) matrices.
A 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is called *invertible* if there exists another 2×2 matrix B such that

$$AB = I \quad \text{and} \quad BA = I,$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the 2×2 identity matrix.
In this case B is unique and we write $B = A^{-1}$, so

$$AA^{-1} = A^{-1}A = I.$$

If A is invertible and $Ax = b$, then

$$x = A^{-1}b.$$

20.2 Finding A^{-1} for a 2×2 matrix

For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the determinant of A is

$$\det(A) = ad - bc.$$

If $\det(A) \neq 0$, then A is invertible and the inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

You can also find A^{-1} using Gaussian elimination on the augmented matrix

$$[A \mid I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right].$$

Apply row operations until the left side becomes I:

$$[A \mid I] \longrightarrow [I \mid A^{-1}].$$

Warning: When the inverse does not exist

If, when you row-reduce $[A \mid I]$, the left-hand side cannot be turned into the identity matrix (for example, you get a row of zeros on the left), then A is *not* invertible.
For a 2×2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

another way to see this is using the determinant:

$$\det(A) = ad - bc = 0 \quad \Longrightarrow \quad A \text{ is not invertible.}$$

In this situation we say that A is *singular* and it has no inverse.

Practice: Unit 20 Practice

1. Let
$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$
Compute $\det(A)$ and then find A^{-1} using the 2×2 formula.

2. Check your answer by computing both AA^{-1} and $A^{-1}A$ and verifying that each product is the identity matrix.

3. Give an example of a 2×2 matrix B that is singular (not invertible). Show that $\det(B) = 0$ and explain, using row-reduction on $[B \mid I]$, why you cannot turn the left side into the identity.